IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

Reduced phase spaces of models of collective motion

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1988 J. Phys. A: Math. Gen. 21 1805

(http://iopscience.iop.org/0305-4470/21/8/016)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 129.252.86.83
The article was downloaded on 01/06/2010 at 06:38

Please note that terms and conditions apply.

J. Phys. A: Math. Gen. 21 (1988) 1805-1810. Printed in the UK

Reduced phase spaces of models of collective motion

G John

Institut für Theoretische Physik, Universität Tübingen, D-7400 Tübingen, Federal Republic of Germany

Received 1 December 1986, in final form 23 October 1987

Abstract. The phase spaces $T^*\mathrm{GL}^+(3,\mathbb{R})$ and $T^*\mathrm{SL}(3,\mathbb{R})$, which underlie the dynamics of certain models of collective motion, are reduced with respect to the symmetry group of spatial rotations. It is shown that the resulting reduced phase spaces are generated by the orbits of Hamiltonian group actions of the groups $\mathrm{GCM}(3)=\mathrm{M}^s(3)\,\textcircled{s}\,\mathrm{GL}^+(3,\mathbb{R})$ or $\mathrm{CM}(3)=\mathrm{M}^s(3)\,\textcircled{s}\,\mathrm{SL}(3,\mathbb{R})$ on the cotangent bundles.

Generalised phase spaces have become very powerful tools in understanding properties of dynamical systems [1, 2]. On the other hand it has been well known since the early days of quantum mechanics that the canonical structure of phase space provides a link between classical mechanics and quantum mechanics which some time ago was reformulated in the so-called geometric quantisation. This method is also useful for understanding models of collective motion [3, 4]. These systems of collective motion, which we are going to consider, will be assumed to be invariant with respect to rotations in space. Consequently we will be able to reduce the number of degrees of freedom by constructing reduced phase spaces using the Kostant-Souriau-Sternberg reduction procedure.

The notation which will be used in this paper is somewhat different from that which is customary in differential geometry. Let x denote elements of the manifold. Vector fields will be denoted by symbols like δx, $\delta' x$ (cf [5]). If the manifold has the structure of a Lie group vector fields can be mapped into the tangent space at the neutral element e by left or right multiplication. The image of a vector field δa will be denoted by $\delta^{\mathrm{l}} a = a^{-1}\delta a$ or by $\delta^{\mathrm{r}} a = (\delta a)a^{-1}$.

We will only be interested in Lie groups which are subgroups of the general linear group $\mathrm{GL}(N,\mathbb{R})$ for some finite N (whereby in physics $N=3$). Here a dual pairing of tangent spaces is given by

$$\langle A, B\rangle = \mathrm{Tr}\, A^{\mathrm{t}}B \qquad (1)$$

where A and B are elements of the real linear space $\mathrm{M}(N)$ of $N\times N$ matrices and A^{t} denotes the transpose of the matrix A. The pairing will be used to define an explicit representation of differential forms on linear groups [6]. For Lie subgroups of $\mathrm{GL}(N,\mathbb{R})$ the Lie algebra is represented by linear subspaces of $\mathrm{M}(N)$. The dual space according to the pairing (1) consists of classes of matrices which are linear submanifolds of $\mathrm{M}(N)$. Indeed matrices which belong to the same class differ by an element of the annihilator of the Lie algebra. It is convenient to denote a class by a certain element of the corresponding submanifold.

 © 1988 IOP Publishing Ltd

Proposition 1. Let G be a linear group, g_l its Lie algebra of left-invariant vector fields and g_l^* the dual space of left-invariant linear forms. The cotangent bundle T^*G is homeomorphic to $G \times g_l^*$. Let $x = (a, \rho)$ denote a point of the cotangent bundle. Then its canonical 1-form is given by

$$\theta(\delta x) = \mathrm{Tr}\, \rho^t \delta a^l \qquad \delta a^l \in g_l \qquad \rho \in g_l^*$$

and the corresponding symplectic form is

$$\sigma(\delta x, \delta' x) = \mathrm{Tr}((\delta'^l a)^t \delta\rho - (\delta'\rho)^t \delta a^l - \rho^t[\delta' a^l, \delta a^l]). \tag{2}$$

Similarly if one uses the Lie algebra of right-invariant fields g_r, its dual space g_r^* and coordinates (a, λ) one obtains

$$\theta(\delta x) = \mathrm{Tr}\, \lambda^t \delta a^r \qquad \delta a^r \in g_r \qquad \lambda \in g_r^*$$

and the symplectic form is given by

$$\sigma(\delta x, \delta' x) = \mathrm{Tr}((\delta' a^r)^t \delta\lambda - (\delta'\lambda)^t \delta a^r + \lambda^t[\delta' a^r, \delta a^r]). \tag{3}$$

The proof of this proposition is an immediate consequence of the notation. Note that if appropriate representatives of the matrices ρ and λ are chosen, they are related by $\lambda = -(a^t)^{-1}\rho a^t$. In the case of the rigid body the pair (a, ρ) is related to body coordinates and the pair (a, λ) to space coordinates.

Both right multiplications $a \to aA^{-1}$ and left multiplications $a \to Aa$ extend to Hamiltonian actions on the cotangent bundle. They are

$$(a, \rho) \to (aA^{-1}, (A^{-1})^t \rho A^t) \qquad \text{or} \qquad (a, \lambda) \to (aA^{-1}, \lambda) \tag{4}$$

and

$$(a, \rho) \to (Aa, \rho) \qquad \text{or} \qquad (a, \lambda) \to (a, (A^t)^{-1} \lambda A^t) \tag{5}$$

respectively and they induce moment maps [1]:

$$J_r(a, \rho) = \rho \qquad J_l(a, \lambda) = \lambda \tag{6}$$

respectively.

The phase space of a system of several particles whose dynamics are restricted to collective rotations and collective oscillations is represented by the cotangent bundle $T^*GL^+(3, \mathbb{R})$ of the general linear group with positive determinant. If additionally the volume is assumed to be constant ('incompressible fluid') the basic group becomes $SL(3, \mathbb{R})$ (cf [3]). This can be shown in an elementary way. One demands that the motion of all particles is restricted to one of these groups according to the physical situation one wants to consider.

The following is an immediate consequence of the direct product structure of the cotangent bundle of an arbitrary Lie group. Let us assume that the dynamical system remains invariant with respect to those Hamiltonian transformations which are induced by left actions of the group on itself. Then the reciprocal image of the fixed momentum of this action is spanned by orbits of Hamiltonian transformations which are induced by right actions of the group on itself. Consequently the reduced phase space which corresponds to this symmetry is given by the image of the moment map of this action [1]. Of course all this is also true, if the words 'right' and 'left' are exchanged.

This reduction applies in particular to the model of the free rigid body. The basic group is the orthogonal group $SO(3, \mathbb{R})$ and the system also remains invariant under rotations of space, i.e. under left actions of $SO(3, \mathbb{R})$ on itself. For elastic bodies the situation is different. The symmetry group is still the rotation group acting on the cotangent bundle by left multiplications but the Lie group of the phase space is $GL^+(3, \mathbb{R})$ or $SL(3, \mathbb{R})$, i.e. the symmetry group is a subgroup of the basic Lie group. The momentum mapping of this symmetry group can be chosen to be

$$\tilde{J}_l(a, \lambda) = \tilde{\lambda} = \tfrac{1}{2}(\lambda - \lambda^t) \tag{7}$$

where we have taken advantage of the above-mentioned freedom, that the representative matrices of the image of the momentum mapping can be chosen such that arbitrary elements of the annihilator can still be added. Here in three dimensions $\tilde{\lambda}$ corresponds to the vector of angular momentum.

Let us assume that $\tilde{\lambda}$ is a regular value of the momentum mapping, so that the inverse image $M_{\tilde{\lambda}}$ of $\tilde{\lambda}$ is a manifold. Of course for each point of the inverse image the orbit of right actions of the whole group $GL^+(3, \mathbb{R})$ (or $SL(3, \mathbb{R})$), which passes this point, is contained in the inverse image. It is not, however, the whole level set since points of the cotangent bundle, whose coordinates λ differ only by symmetric matrices, are mapped to the same angular momentum. It will be shown shortly that these symmetric matrices can be combined with the whole group, such that the inverse image of fixed angular momentum becomes the orbit of a Hamiltonian transformation group on the cotangent bundle.

Let s be an arbitrary element of the linear space $M^s(N)$ $(N = 3)$ of symmetric matrices. The transformations

$$s \to (B^t)^{-1} s B^{-1} =: \boldsymbol{B}s \qquad B \in GL^+(N, \mathbb{R})$$

define a linear representation of the general linear group. Define functions

$$f_s^W(a) := \operatorname{Tr} W(a^t)^{-1} s a^{-1} \qquad W \in M^s(N)$$

on $GL^+(N, \mathbb{R})$ (cf [3]). Right action of the group on itself implies

$$((r_{B^{-1}})^* f_s^W)(a) = f_s^W(aB) = f_{\boldsymbol{B}s}^W(a). \tag{8}$$

Being pulled back to the cotangent bundle these functions define Hamiltonian vector fields

$$\delta x = (\delta a, \delta \rho) = (0, -2sa^{-1} W(a^t)^{-1})$$

which integrate to actions of the Abelian group $M^s(N)$

$$(a, \rho) \to (a, \rho - Sa^{-1} W(a^t)^{-1}). \tag{9}$$

Combining this group action with the extended right action leads to a Hamiltonian action of a semidirect product group:

$$GCM(N) := M^s(N) \circledS GL^+(N, \mathbb{R})$$

with multiplication law

$$(S, A)(T, B) = (S + (A^t)^{-1} T A^{-1}, AB) \qquad S, T \in M^s(N);\ A, B \in GL^+(N, \mathbb{R})$$

on the cotangent bundle. We call GCM(3) the generalised collective motion group. Its action on $T^*GL(3, \mathbb{R})$ is explicitly given by

$$(S, A)(a, \rho) = (aA^{-1}, (A^t)^{-1} \rho A^t - SAa^{-1} W(a^t)^{-1} A^t). \tag{10}$$

Note that relation (8) is crucial for defining this semidirect product action (cf [3]).

Proposition 2. Let $\tilde{J}_1: T^*\mathrm{GL}^+(3,\mathbb{R}) \to \mathrm{so}(3,\mathbb{R})^*$ denote the moment map which corresponds to left actions of the rotation group on the cotangent bundle of the general linear group, and let λ_0 be an element of the image. The inverse image of the moment map is spanned by the free Hamiltonian group action of GCM(3) through the point $(a,\rho)=(e,-\lambda_0)$ (e denotes the neutral element of the group). In (a,ρ) coordinates this orbit is given by

$$(S,A)(e,-\tilde{\lambda}_0)=(A^{-1},-(A^{\mathrm{t}})^{-1}\tilde{\lambda}_0 A^{\mathrm{t}}-SAA^{\mathrm{t}}) \qquad A\in \mathrm{GL}^+(3,\mathbb{R});\ S\in \mathrm{M}^s(3). \tag{11}$$

Relation (11) is a special case of (10) in which W is equal to the unit matrix. That this is a necessary condition follows from the fact that the actions of GCM(3) and of the rotation group must commute. It follows immediately from the relation between right-invariant momentum λ and left-invariant momentum ρ that the orbit is mapped by the moment map to $\tilde{\lambda}_0$. That the action is free follows by direct computation.

Considered as a submanifold of the cotangent bundle the inverse image of fixed angular momentum is not yet a reduced phase space, since the reduced canonical 2-form is degenerate. From general theorems it follows that the kernel of this presymplectic form coincides with the kernel of the moment map of the symmetry group, i.e. of the rotation group. Basically this allows us to construct the reduced phase space. But here also, as in the well known model of the rigid body, everything becomes more explicit if use is made of the fact that level sets of constant angular momentum coincide with orbits of a certain group of Hamiltonian transformations on the phase space and that again the moment map can be applied.

Proposition 3. Let gcm(3) be the Lie algebra of the generalised collective motion group and gcm(3)* its dual space, the elements of which are represented by pairs

$$m=(\nu,\mu) \qquad \nu\in \mathrm{M}^s(3) \qquad \mu\in \mathrm{M}(3).$$

The moment map of the action GCM(3) on the cotangent bundle is given by

$$J':(a,\rho)\to m=(\nu,\mu)=(\tfrac{1}{2}(a^{\mathrm{t}}a)^{-1},\rho). \tag{12}$$

There is a theorem (Kostant-Souriau theorem) (cf [1, 5, 7]) which states that by the moment map orbits of canonical transformation groups are mapped onto orbits of the coadjoint action in the dual space of the Lie algebra (up to a certain first cohomology class which in our case vanishes).

For calculating orbits of the coadjoint action it is helpful to represent GCM(3) as a subgroup of $\mathrm{Sp}(6,\mathbb{R})$ setting

$$(S,A)=\begin{pmatrix} A & 0 \\ SA & A^{\mathrm{t}-1}\end{pmatrix} \qquad A\in \mathrm{GL}(3,\mathbb{R});\ S\in \mathrm{M}^s(3). \tag{13}$$

In particular, this means that the collective motion group is a subgroup of the linear group and we can apply again those techniques which are described in [6]. Elements of the Lie algebra gcm(3) are represented by

$$\gamma=\begin{pmatrix} \xi & 0 \\ \eta & -\xi^{\mathrm{t}}\end{pmatrix} \qquad \xi\in \mathrm{M}(3);\ \eta\in \mathrm{M}^s(3).$$

Using the dual pairing (1) for linear groups the elements of gcm(3)* are represented by

$$m=\begin{pmatrix} \frac{1}{2} & 0 \\ \nu & -\frac{1}{2}\mu\end{pmatrix} \qquad \mu\in \mathrm{M}(3);\ \nu\in \mathrm{M}^s(3)$$

and we obtain for the coadjoint representation

$$\mathrm{ad}^*_{(S,A)}(\nu, \mu) = (A\nu A^{t}, (A^{t})^{-1}\mu A^{t} - SA\nu A^{t}). \tag{14}$$

Incidentally a comparison of (11) and (14) shows us that the momentum mapping is indeed equivariant.

There are essentially two types of orbits. For $\tilde{\lambda}_0 \neq 0$ the isotropy group at $(\frac{1}{2}I, -\tilde{\lambda}_0)$ (I = unit matrix) is the group SO(2) of those orthogonal transformations which leave $\tilde{\lambda}_0$ invariant. The dimension of the reduced phase space is $15-1=14$. For $\tilde{\lambda}_0 = 0$ the isotropy group at $(\frac{1}{2}I, 0)$ is isomorphic to SO(3) and the dimension of the reduced phase space is $15-3=12$.

On the coadjoint orbits of GCM(3) the symplectic form is obtained from

$$\sigma(\delta m, \delta' m) = \langle m, [\delta(A, S), \delta'(A, S)]\rangle \tag{15}$$

where $\langle\ ,\ \rangle$ means the dual pairing and $\delta(A, S)$, $\delta'(A, S)$ are tangent vectors. They will be characterised by elements of the Lie algebra gcm(3). The letter m denotes an element of gcm(3)*. This leads to

$$\sigma(\delta m, \delta' m) = \mathrm{Tr}(AA^{t}(\eta\xi' - \eta'\xi) - (A\rho_0 A^{-1} + AA^{t}S)[\xi, \xi']). \tag{16}$$

In the case of the so-called liquid-drop model with phase space $T^*\mathrm{SL}(3)$ the group GCM(3) has to be substituted by the collective motion group $\mathrm{CM}(3) = \mathrm{M}^s(3) \circledS \mathrm{SL}(3, \mathbb{R})$. This group can also be represented by (13). In the Lie algebra tangent vectors of $\mathrm{SL}(3, \mathbb{R})$ are represented by traceless matrices and the same can be done by left-invariant and right-invariant momenta. Therefore we substitute

$$\rho - \tfrac{1}{3}I\,\mathrm{Tr} \to \rho \qquad I = \text{unit matrix}.$$

The Hamiltonian action CM(3) on $T^*\mathrm{SL}(3, \mathbb{R})$ is given by

$$(S, A)(a, \rho) = (aA^{-1}, (A^{t})^{-1}\rho A^{t} - SA(a^{t}a)^{-1}A^{t} + \tfrac{1}{3}I\,\mathrm{Tr}\, SA(a^{t}a)^{-1}A^{t})$$

and the corresponding moment map is again given by (12), namely by

$$J' : (a, \rho) \to (\nu, \mu) = (\tfrac{1}{2}(a^{t}a)^{-1}, \rho). \tag{17}$$

The coadjoint representation is

$$\mathrm{ad}^*_{(S,A)}(\nu, \mu) = (A\nu A^{t}, (A^{t})^{-1}\mu A^{t} - 2SA\nu A^{t} + \tfrac{2}{3}I\,\mathrm{Tr}(SA\nu A^{t}). \tag{18}$$

The actions of these mappings differ from those of the elastic body insofar as CM(3) no longer acts freely. At the point (e, ρ_0) the isotropy group of the CM(3) action is given by pairs (I, sI) with real parameter s. On the other hand, at $m = (\frac{1}{2}I, -\lambda_0)$ the isotropy group consists of pairs (sI, r) where r denotes those rotations which leave λ_0 fixed. This group is isomorphic to $\mathbb{R} \times \mathrm{SO}(2)$.

Apart from this difference everything is analogous to the situation described above. Surfaces of constant angular momentum are again spanned by orbits of CM(3). Note that the kernel of the Hamiltonian action on the phase space coincides with the non-compact part of the isotropy group of the coadjoint action.

In some articles the group CM(3) has been compared with a similar semidirect product which has been constructed from the rotation group SO(3) and the linear space of traceless symmetric matrices $\mathrm{M}^s_0(3)$. This group is related to the rigid body. The action of this rigid body group, sometimes called RB(3), is transitive on the phase space $T^*\mathrm{SO}(3)$. This is a situation which is different from that of non-rigid bodies, for here the group RB(3) does not generate reduced phase spaces. In fact it represents the complete phase space $T^*\mathrm{SO}(3)$ as a homogeneous space.

The construction of reduced phase spaces can be a preliminary step in solving the problem to quantise a Hamiltonian system. Here it is a special advantage if these reduced phase spaces are spanned by certain group actions, for it opens the possibility of quantising the system by constructing unitary representations of groups (cf [8]).

References

[1] Abraham R and Marsden J E 1978 *Foundations of Mechanics* (Reading, MA: Benjamin)
[2] Arnold V I 1978 *Mathematical Methods of Classical Mechanics* (Berlin: Springer)
[3] Guillemin V and Sternberg S 1980 *Ann. Phys., NY* **127** 220-53
[4] Rosensteel G and Ihrig E 1979 *Ann. Phys., NY* **121** 113-30
[5] Souriau J M 1970 *Structure des Systèmes Dynamiques* (Paris: Dunod)
[6] Kirillov A A 1970 *Elements of the Theory of Representations* (Berlin: Springer)
[7] Marle G M 1976 *Differential Geometry and Relativity* ed M Cohen and M Flato (Dordrecht: Reidel)
[8] Souriau J M 1976 *Lecture Notes in Mathematics* vol 570 (Berlin: Springer)